AN EXACT SOLUTION TO AN ELASTIC-PLASTIC STRESS-CONCENTRATION PROBLEM (*)

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Abstract. An exact elastic-plastic solution is found for the stresses around a circular hole in a sheet subjected to balanced biaxial tension at infinity. The sheet is orthotropic, but isotropic in its plane. The solution, based on a J_2 type of deformation theory together with a modified Ramberg-Osgood law, is rigorously valid for flow theory as well.

Introduction. In 1946 Iliushin [1] stated his celebrated theorem concerning certain sufficient conditions for a linear dependence of stress on loading in a boundary-value problem of plasticity. Suppose that the traction is specified on all boundaries, with its spatial distribution invariant but its magnitude monotonically increasing. Iliushin observed that the stresses in the body vary linearly with the magnitude of the loading if the problem is solved on the basis of the simple J_2 deformation theory of plasticity in conjunction with a pure-power law $\varepsilon = k\sigma^n$ for the uniaxial stress-strain relation. Under these circumstances, the solution will also be rigorously valid for J_2 flow theory because "proportional loading" – constant ratios of the stress components – exist at each point. Finally, it may be noted that in a stress-concentration problem, the stress-concentration factor would be independent of the loading magnitude; indeed, it would only depend on the strain-hardening exponent n. It is evident that the pure-power law corresponds to a rigid-plastic material rather than an elastic-plastic one, such as that which would be described by the uniaxial Ramberg-Osgood relation $\varepsilon = \sigma E^{-1} + k\sigma^n$. In this latter case a stress-concentration factor would start out with its elastic value and approach the pure-power-law value as the loading increased. There would be no reason to expect proportional loading in the interior of the body, and hence the J_2 flow and deformation theories could be presumed to give different results.

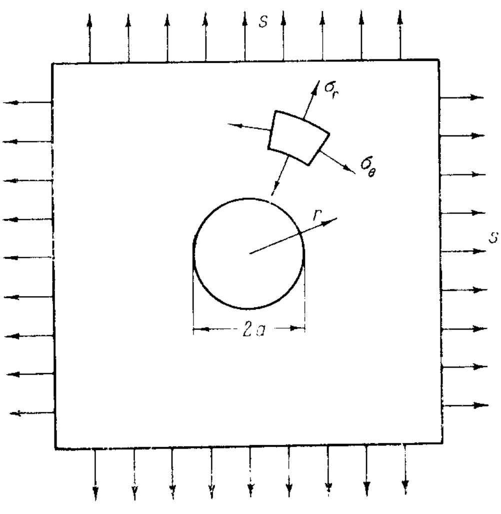

Fig. 1

The plane stress problem of a circular hole in an infinite sheet subjected to balanced

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biaxial tension S at infinity (Fig. 1) was studied in [2] on the basis of J_2 deformation theory and the Ramberg-Osgood law. In this work a very useful manipulation which eliminated the space variable facilitated the solution. The stress-concentration factor $K=\sigma_\theta/S$ hole was found analytically for the pure-power-law case, corresponding to $S\to\infty$; the transition from the elastic value $K=2$ to the limiting power-law value was determined numerically. As expected, there was not proportional loading in the interior of the sheet.

The present paper is a sequel to paper [2]. Instead of the Ramberg-Osgood law, a slightly modified uniaxial relation of the form

$$\varepsilon=\frac{\sigma}{E}\quad\text{for }|\sigma|\leqslant\sigma_y,\qquad \varepsilon=\frac{\sigma}{E}\left|\frac{\sigma}{\sigma_y}\right|^{n-1}\quad\text{for }|\sigma|\geqslant\sigma_y \tag{1}$$

is assumed. It turns out that with this modification an exact analytic solution for K over the full range of S becomes possible on the basis of J_2 deformation theory (*). Furthermore, as will be shown, the solution is then also exact for J_2 flow theory. This remarkable situation occurs because, even though the requirement of proportional loading is not met, it is violated only while stresses are in the elastic range.

The problem of [2] is generalized in this paper by consideration of materials that are orthotropic (as in [3]), but remain isotropic in the plane of the sheet. With the use of appropriately modified elastic-plastic stress-strain relations of the J_2 type, exact solutions are still obtained for all S, and the coincidence of flow and deformation theories continues to hold (**).

Stress-strain relations. The plastic orthotropy of the sheet is conventionally characterized by the parameter R, defined on the basis of a uniaxial, in-plane tension test as the ratio of the transverse plastic strain in the plane of the sheet to the plastic strain through the thickness. In the present plane stress problem, only the radial stress σ_r and circumferential stress σ_θ are involved, and the corresponding plastic strain rates ε_r^p and ε_θ^p would be given by Hill's [5] generalized flow theory as

$$\begin{aligned}\dot{\varepsilon}_r^{p}&=\left(\frac{1}{E_t}-\frac{1}{E}\right)\left(\sigma_r-\frac{R}{1+R}\sigma_\theta\right)\left(\frac{\dot{\sigma}}{\sigma}\right)\\ \dot{\varepsilon}_\theta^{p}&=\left(\frac{1}{E_t}-\frac{1}{E}\right)\left(\sigma_\theta-\frac{R}{1+R}\sigma_r\right)\left(\frac{\dot{\sigma}}{\sigma}\right)\end{aligned} \tag{2}$$

where σ is the effective stress defined by

$$\sigma=\left(\sigma_r^2+\sigma_\theta^2-\frac{2R}{1+R}\sigma_r\sigma_\theta\right)^{1/2} \tag{3}$$

and E_t is the tangent modulus of the uniaxial stress-strain curve at the stress value σ. The initial yield surface is defined by the ellipse $\sigma=\sigma_y$. For proportional loading, Eqs. (2) can be integrated to give the deformation theory form

*) That this would happen was suggested to the writer over ten years ago by Dr. J. M. Hedgepeth, and was verified by calculations made by Dr. O. M. Mangasarian and the writer at that time.

**) For the pure-power-law case, this orthotropic problem was recently solved by Yang [4] who exploited the same trick used in [2] and [3] of eliminating the space variable.

$$\varepsilon_r{}^p = \left(\frac{1}{E_s} - \frac{1}{E}\right)\left(\sigma_r - \frac{R}{1+R}\sigma_\theta\right)$$
$$\varepsilon_\theta{}^p = \left(\frac{1}{E_s} - \frac{1}{E}\right)\left(\sigma_\theta - \frac{R}{1+R}\sigma_r\right) \quad (4)$$

where E_s is the secant modulus on the uniaxial stress-strain curve at σ. For future reference, note that the integration is valid as long as σ_r and σ_θ follow a radial line through the origin in the stress space when $\sigma \geqslant \sigma_y$; it obviously doesn't matter if the stress path is curved when σ_r and σ_θ are inside the initial surface.

The elastic strains are

$$\varepsilon_r{}^E = \frac{1}{E}(\sigma_r - \nu\sigma_\theta), \qquad \varepsilon_\theta{}^E = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \quad (5)$$

where ν is the Poisson's ratio in the plane of the sheet. But now invoke the extended Michell theorem [6] which says that the stresses in the present problem do not depend on ν. So choose $\nu = R / (1 + R)$, and then the total strains follow from (4) and (5) as

$$\varepsilon_r = \varepsilon_r{}^E + \varepsilon_r{}^p = \frac{1}{E_s}\left(\sigma_r - \frac{R}{1+R}\sigma_\theta\right)$$
$$\varepsilon_\theta = \varepsilon_\theta{}^E + \varepsilon_\theta{}^p = \frac{1}{E_s}\left(\sigma_\theta - \frac{R}{1+R}\sigma_r\right) \quad (6)$$

After the stresses are found, with the use of (6), the actual strain could be calculated from (4) and (5) with the use of the true value of Poisson's ratio.

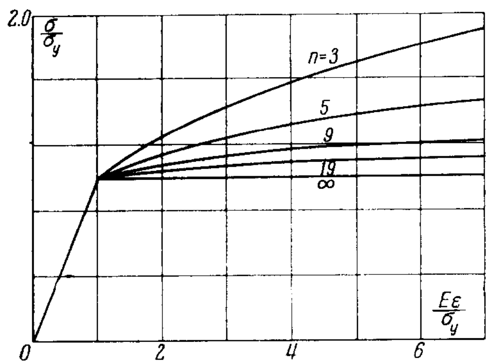


Fig. 2

If the uniaxial relation (1) – shown nondimensionally in Fig. 2 for various values of n – is used, $(1 / E_s)$ is given in terms of the effective stress σ as

$$\frac{1}{E_s} = \frac{1}{E} \quad \text{for} \quad \sigma \leqslant \sigma_y$$
$$\frac{1}{E_s} = \frac{1}{E}\left(\frac{\sigma}{\sigma_y}\right)^{n-1} \quad \text{for} \quad \sigma \geqslant \sigma_y \quad (7)$$

Boundary value problem and solution. At infinity, we suppose that $\sigma_r = \sigma_\theta = s$; at the hole, where $r = a$, we have $\sigma_r = 0$. In terms of the parameter $\lambda = s / \sigma_y$, the solution exhibits three regimes.

i) $0 \leqslant \lambda \leqslant {}^1/_2$.

The well-known elastic solution for the stresses is

$$\sigma_\theta = s\,[1 + (a / r)^2], \quad \sigma_r = s\,[1 - (a / r)^2] \quad (8)$$

It is easily verified that σ takes on its largest value, $2S$, at $r = a$, so that (8) is valid for $S \leqslant {}^1/_2\sigma_y$, and the stress-concentration factor

$$K \equiv \sigma_\theta\,(a) / s = 2 \quad \text{for} \quad \lambda \leqslant {}^1/_2$$

ii) ${}^1/_2 \leqslant \lambda \leqslant \sqrt{{}^1/_2(1 + R)}$.

In this range, the effective stress σ is less than σ_y at infinity, and so the stresses stay in the elastic range for r larger than some critical value r_*, which depends on λ. In this elastic domain, the solution for the stresses can obviously be written in the form

$$\sigma_\theta / \sigma_y = \lambda [1 + \beta(r_* / r)^2], \quad \sigma_r / \sigma_y = \lambda [1 - \beta (r_*/r)^2] \tag{9}$$

where β is a constant that depends on λ. At $r = r_*$, equilibrium requires continuity of σ_r and continuity of the radial displacement $u = r\varepsilon_\theta$ implies continuity of ε_θ. It follows, then, from (6), that σ_θ and σ are also continuous, and so σ, as calculated from (9) at $r = r_*$, must equal σ_y. This provides the value

$$\beta = \frac{1}{\sqrt{1+2R}} \left(\frac{1+R}{2\lambda^2} - 1 \right)^{1/2} \tag{10}$$

and only r_* remains to be determined as a function of λ.

In the plastic domain $r \leqslant r_*$, the solution is effected as follows. Combining (as in [2]) the equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \tag{11}$$

and the compatibility equation

$$d\varepsilon_\theta / dr + (\varepsilon_\theta - \varepsilon_r) / r = 0 \tag{12}$$

gives

$$\frac{d\sigma_r}{\sigma_\theta - \sigma_r} = \frac{d\varepsilon_\theta}{\varepsilon_r - \varepsilon_\theta} \tag{13}$$

which does not contain r explicitly. Introducing the stress-strain relation (6) (as in [3]) leads to

$$\left(\frac{1}{E}\right)_s^{-1} \frac{d}{d\sigma}\left(\frac{1}{E_s}\right)\left(\sigma_\theta - \frac{R}{1+R}\sigma_r\right) d\sigma + d(\sigma_\theta + \sigma_r) = 0 \tag{14}$$

The substitution

$$\begin{aligned} \sigma_\theta &= \frac{\sigma}{2}\sqrt{2+2R}\left(\cos\alpha + \frac{1}{\sqrt{1+2R}}\sin\alpha\right) \\ \sigma_r &= \frac{\sigma}{2}\sqrt{2+2R}\left(\cos\alpha - \frac{1}{\sqrt{1+2R}}\sin\alpha\right) \end{aligned} \tag{15}$$

in terms of the parameter α satisfies (3) identically, and the fact that, by (7),

$$\left(\frac{1}{E_s}\right)^{-1} \sigma \frac{d}{d\sigma}\left(\frac{1}{E_s}\right) = n - 1$$

for $\sigma \geqslant \sigma_y$, reduces (14) to

$$\begin{aligned} [(n + 1 + 2R)\cos\alpha + (n-1)\sqrt{1+2R}\sin\alpha]\, d\sigma &= \\ = (2 + 2R)\sigma \sin\alpha d\alpha \end{aligned} \tag{16}$$

in the plastic domain.

At $r = r_*$, α takes on the value α_* determined by the requirement that (15), with $\sigma = \sigma_y$, agree with (9). Thus,

$$\alpha_* = \operatorname{arc\,tg}\sqrt{{}^1/_2(1+R)\lambda^{-2} - 1} \tag{17}$$

Integrating the differential equation (16), with the initial condition $\sigma = \sigma_y$ at $\alpha = \alpha_*$, then gives

$$\frac{\sigma}{\sigma_y} = \left(\frac{n+1+2R+(n-1)\sqrt{1+2R}\sqrt{{}^1/_2(1+R)\lambda^{-2}-1}}{\sqrt{{}^1/_2(1+R)\lambda^{-2}}\,[(n+1+2R)\cos\alpha + (n-1)\sqrt{1+2R}\sin\alpha]}\right)^{\mu}\times$$

$$\times\exp\left(\frac{(n-1)\sqrt{1+2R}}{n^2+1+2R}(\alpha - \alpha_*)\right) \qquad \left(\mu = \frac{n+1+2R}{n^2+1+2R}\right) \tag{18}$$

At the hole, $\sigma_r = 0$ and hence, by (15), α at the hole takes on the value

$$\alpha_a = \operatorname{arc\,tg} \sqrt{1+2R} \qquad (19)$$

Hence, the stress concentration factor is

$$K = \frac{1}{\lambda}\left(\frac{(n+1+2R)+(n-1)\sqrt{1+2R}\sqrt{{}^1\!/_2(1+R)\lambda^{-2}-1}}{n(1+R)\lambda^{-1}}\right)^{\mu}\times$$

$$\times \exp\left(\frac{(n-1)\sqrt{1+2R}}{n^2+1+2R}(\alpha_a-\alpha_*)\right) \qquad (20)$$

for ${}^1\!/_2 \leqslant \lambda \leqslant \sqrt{{}^1\!/_2(1+R)}$. Combining (18) and (20) provides an alternative expression for σ in the form

$$\frac{\sigma}{\sigma_y} = \lambda K\left(\frac{n\sqrt{2+2R}}{(n+1+2R)\cos\alpha+(n-1)\sqrt{1+2R}\sin\alpha}\right)^{\mu}\exp\left(\frac{(n-1)\sqrt{1+2R}}{n^2+1+2R}(\alpha-\alpha_a)\right) \qquad (21)$$

The stresses in the plastic domain are now given by Eqs. (15), with $\alpha_* \leqslant \alpha \leqslant \alpha_a$, and σ given by (21). We must, however, relate r to α in the range $r_* \geqslant r \geqslant a$ in order to find the spatial distribution of the stresses.

From the equilibrium equation (11) and the use of (15)

$$\frac{2dr}{r} = -\sqrt{1+2R}\,d\alpha - \frac{d\sin\alpha}{\sin\alpha} + [\sqrt{1+2R}\operatorname{ctg}\alpha - 1]\frac{d\sigma}{\sigma}$$

Elimination of $\sigma^{-1}d\sigma$ with the help of (16) provides a first order differential equation in r and α which can be integrated, with the initial condition $\alpha = \alpha_a$ at $r = a$ to give

$$\frac{r}{a} = \left(\frac{1+2R}{2+2R}\right)^{1/4}(\csc\alpha)^{1/2}\left(\frac{(n+1+2R)\cos\alpha+(n-1)\sqrt{1+2R}\sin\alpha}{n\sqrt{2+2R}}\right)^{\gamma}\times \qquad (22)$$

$$\times\exp\left(\frac{(n^2-1)\sqrt{1+2R}}{2(n^2+1+2R)}(\alpha_a-\alpha)\right) \qquad \left(\gamma = \frac{n(1+R)}{n^2+1+2R}\right)$$

This is valid for $\alpha_* \leqslant \alpha \leqslant \alpha_a$. It is remarkable that r does not depend on λ. This means (see (15)) that once the stress state at any point in the sheet enters the plastic range, the ratio σ_θ/σ_r remains constant. In other words, proportional loading occurs for $\sigma \geqslant \sigma_y$.

The location r_* of the elastic-plastic boundary, found by setting $\alpha = \alpha_*$ in (22), is given by

$$\frac{r_*}{a} = \left(\frac{1+2R}{2(1+R-2\lambda^2)}\right)^{1/4}\left(\frac{n+1+2R+(n-1)\sqrt{1+2R}\sqrt{{}^1\!/_2(1+R)\lambda^{-2}-1}}{n(1+R)\lambda^{-1}}\right)^{\gamma}\times$$

$$\times\exp\left(\frac{(n^2-1)\sqrt{1+2R}}{2(n^2+1+2R)}(\alpha_a-\alpha_*)\right) \qquad \left({}^1\!/_2 \leqslant \lambda \leqslant \sqrt{{}^1\!/_2(1+R)}\right) \qquad (23)$$

iii) $\lambda \geqslant \sqrt{{}^1\!/_2(1+R)}$.

Now the stresses are everywhere in the plastic range, and the integration of (16) is executed with the initial condition $\alpha = 0$, $\sigma = s\sqrt{2/(1+R)}$ implied by $\sigma_\theta = \sigma_r = S$ at infinity. The result is

$$\frac{\sigma}{s} = \sqrt{2/(1+R)}\left(\frac{n+1+2R}{(n+1+2R)\cos\alpha+(n-1)\sqrt{1+2R}\sin\alpha}\right)^{\mu}\times$$

$$\times\exp\left(\frac{(n-1)\sqrt{1+2R}}{n^2+1+2R}\,\alpha\right) \tag{24}$$

and then K, found by setting $\alpha = \alpha_a$ in (24), is

$$K = \sqrt{2/(1+R)}\left(\frac{n+1+2R}{n\sqrt{2+2R}}\right)^{\mu}\exp\left(\frac{(n-1)\sqrt{1+2R}}{n^2+1+2R}\,\alpha_a\right) \tag{25}$$

For $R = 1$, this is the same as the pure-power-law result given in [2]; with due regard for notational differences, it checks the pure-power-law result for arbitrary R found in [4].

The formula (22) for r remains valid, but, of course, r_* is now infinite.

Discussion of results. To sum up, the stress-concentration factor K, as a function of $\lambda = s / \sigma_y$ is given by

$K = 2$ for $\lambda \leqslant {}^1/_2$

Eq. (20) for ${}^1/_2 \leqslant \lambda \leqslant ({}^1/_2\,(1+R))^{1/2}$

Eq. (25) for $\lambda \geqslant ({}^1/_2\,(1+R))^{1/2}$

For $R = 1$, these results are plotted in Fig. 3 for several values of n. Note that for $n = \infty$, $K = 1 / \lambda$ for $\lambda > {}^1/_2$, and the curve of K vs. λ must end at $\lambda = 1$, since the applied stress at infinity can not exceed S.

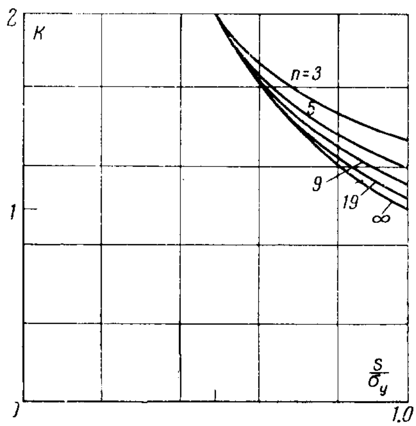


Fig. 3

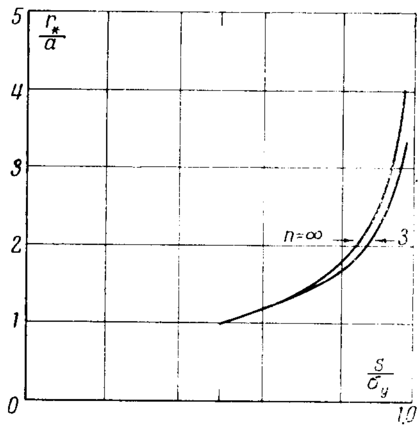
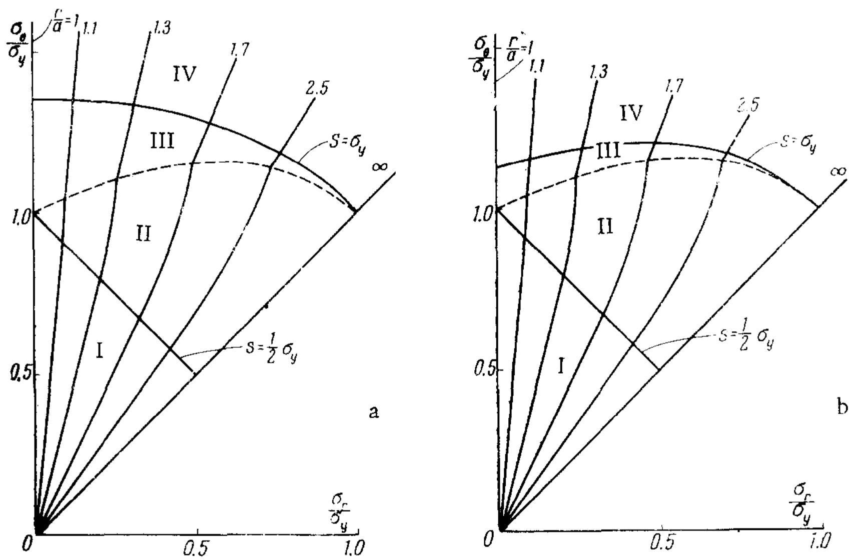


Fig. 4

Numerical calculations show that for $\lambda > {}^1/_2$, K is slowly decreasing function of R. For $R = 1$, the radius r_* of the plastic region, given by Eq. (23), is shown in Fig. 4, as a function of λ, for $n = 3$ and $n = \infty$. Evidently r_* is fairly insensitive to n.

It is amusing to study the stress histories at various values of r / a. Fig. 5a shows how such histories, for $R = 1$ and $n = 3$, at $r / a = 1,\ 1.1,\ 1.3,\ 1.7,\ 2.5$ and ∞, exhibit four regimes. Up to $S = {}^1/_2\,\sigma_y$, the stresses are everywhere in the elastic range, and the radial stress paths are governed by Eqs. (8). The initial regime I in Fig. 3 is bounded by the line $\sigma_\theta / \sigma_y + \sigma_r / \sigma_y = 1$.

As S is further increased, the stresses at any given value of r/a remain elastic (in regime II) until the stress path, governed now by Eqs. (9), hits the Mises yield surface shown by the dotted line. In regime II, the stress paths for points in the interior of the sheet are c u r v e d. In regime III, the sheet is partially plastic, and the stresses are given by (15), where α is given implicitly in terms of r/a by (22). In this regime, the loading is again "proportional" – but the magnitudes of the stresses vary linearly with (λK), as shown by (21), and hence are nonlinear functions of S. Finally, in regime IV, the sheet is everywhere plastic, the stress paths continue on the radial paths of regime III, but now the stresses are proportional to λ. The upshot of all this is that the present results are rigorously correct for J_2 flow theory as well as for J_2 deformation theory, since propor-



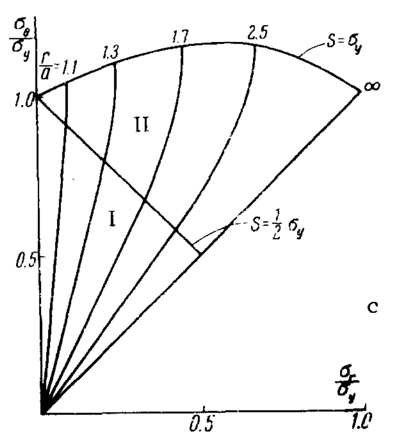


Fig. 5 a, b, c

tional loading occurs at every point in the sheet once σ exceeds σ_y.

A similar set of stress paths is shown in Fig. 5b for $R = 1$, $n = 9$, showing how regime III shrinks in size with increasing n. Finally, Fig. 5c shows the stress paths for $n = \infty$. Here, regimes III and IV collapse into the Mises yield surface, since σ can not exceed σ_y. In all cases, the stress paths have continuous slopes at the interface between regime I and II; in the case $n = \infty$, the stress paths hit the yield surface with vertical slopes for $0 < r / a < \infty$.

Concluding remarks. Similar exact solutions, tied to J_2 deformation theory and the special uniaxial stress-strain curves used herein, can be generated for other problems with radial symmetry; radial stresses applied at the hole, plates of finite radius, and so on. However, the extraordinary validity of the present solution for both J_2 flow theory and J_2 deformation theory need not hold in other cases.

Diagrams.

Fig. 1 Balanced biaxial tension on infinite sheet with hole.

Fig. 2 Nondimensional stress-strain curves.

Fig. 3 Variation of stress-concentration factor with applied stress, $R = 1$.

Fig. 4 Size of plastic region; $R = 1$.

Fig. 5a Stress paths at various values of r/a; $R = 1$, $n = 3$.

Fig. 5b Stress paths at various values of r/a; $R = 1$, $n = 9$.

Fig. 5c Stress paths at various values of r/a ; $R = 1$, $n = \infty$.

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